A Derivation of an Upper Bound for the Number of Configurations of an n×n×n×n Rubik's Cube

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1. Introduction

$C_4(n)$ is a formula for an upper bound of the number of distinguishable configurations of an n×n×n×n Rubik's Cube, which will be derived in this paper. It will be assumed that the reader is familiar with a 4-dimensional Rubik's Cube. Online, one can find the free computer program Magic Cube 4D, developed by Melinda Green, Don Hatch, and Jay Berkenbilt, which is a completely interactive representation of a 4-dimensional Rubik's Cube, and which was the inspiration for this paper and much of my other work.[1] An FAQ page has been provided to help familiarize new users with the necessary concepts of higher dimensions and how Rubik's Cubes would function in these spaces. Additionally, a solution guide has been provided by Roice Nelson, who is another pioneer in the research of higher-dimensional puzzles. His creations include the free programs MagicCube5D, which was written along with Charlie Nevill, and Magic120Cell, which are representations of a 5-dimensional Rubik's Cube and a puzzle based on the 120-cell, respectively.[2,3] I would like to thank Roice in particular for his continual support and encouragement, which includes both hosting this paper and my other work on his website, and proofreading this paper while it was being developed. Roice found many oversights and errors, all of which have been corrected, and provided simplifications and new ideas. His creations MagicCube5D and Magic120Cell have also inspired me, and my work is focused on these programs as well. It should also be mentioned that my discoveries would not have been possible without the previous investigations of H. J. Kamack and T. R. Keane in their paper, "The Rubik Tesseract"; it was used extensively in developing sections 3 and 4 of this paper.[4] Eric Balandraud's article, "Calculating the Permutations of 4D Magic Cubes", was also helpful, and greatly assisted me in examining the properties of 4-dimensional Rubik's Cubes.[5]

2. The Plan

Here is $C_4(n)$:

$$C_4(n) = \frac{15!\cdot 12^{15}}{6}\left(24!\cdot 32!\cdot 2^{26}\cdot 6^{33}\right)^{n\,\mathbf{mod}\,2}\left(\frac{64!}{2}\cdot 3^{63}\right)^{\left\lfloor\frac{n-2}{2}\right\rfloor}\left(\frac{96!}{24^{24}}\cdot 2^{95}\right)^{\left\lfloor\frac{n-2}{2}\right\rfloor+(n\,\mathbf{mod}\,2)\left(\frac{n-3}{2}\right)}\cdot$$

$$\left(\frac{192!}{24^{48}}\right)^{\frac{\left\lfloor\frac{n-4}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor}{2}}\left(\frac{64!}{(8!)^8}\right)^{\left\lfloor\frac{n-2}{2}\right\rfloor}\left(\frac{96!}{(12!)^8}\right)^{(n\,\mathbf{mod}\,2)\left(\frac{n-3}{2}\right)}\left(\frac{48!}{(6!)^8}\right)^{(n\,\mathbf{mod}\,2)\left(\frac{n-3}{2}\right)}\cdot$$

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{\left\lfloor\frac{n-4}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor}{2}+\frac{(n\,\mathbf{mod}\,2)(n-5)(n-3)(n-1)+\left|(n\,\mathbf{mod}\,2)-1\right|(n-4)(n-3)(n-2)}{24}}$$

We will deduce this formula in two stages. First, we will calculate the specific values of $C_4(n)$ for $2 \le n \le 8$. Then we will generalize our findings and build the formula up term by term.

3. The 2^4 Cube

A 2^4 Rubik's Cube consists of 16 corner pieces, each with four stickers (which are also called facelets). There is an important note to be made for cubes regarding the number of pieces per edge. If it is odd (an odd cube), we can fix the central 1-colored pieces in place and observe how the other pieces permute and orient around them. This is valid because the central pieces never move relative to each other, and we can therefore fix them in space. Note also that we never have to consider making a slice move that repositions the central pieces because this is equivalent to rotating all of the other parallel layers in the opposite direction and reorienting the entire cube.

For cubes with an even number of pieces per edge (an even cube), such as the 2^4, we will need a way to fix the cube in space so that we do not inadvertently count extra configurations due to the fact that the entire cube can rotate in 4-space. This can be accomplished by fixing a corner piece in place, and using it as a point of reference in the same manner as the central pieces for an odd cube.

Now, back to the 2^4 cube. We count $2^4 = 16$ pieces, all of which are corners with four facelets each. To determine the number of permutations the corner pieces can attain, we examine what takes place when we make a 90 degree face rotation. Observe that we never need to consider other types of rotations because any possible face rotation can be represented as a sequence of 90 degree face rotations. In such a rotation, 8 corner pieces change position in the form of two 4-cycles, which is an even permutation. Thus the number of ways the corners can be permuted is 16!/2, dividing by two because of the even parity. However, remember that we must fix one corner in place, making the count 15!/2.

To determine the number of ways the corners can be oriented, we restate here the method discovered by T. R. Keane and H. J. Kamack, and described in their paper, "The Rubik Tesseract". This is the only section in this paper which makes use of basic group theory concepts.

In their paper, Keane and Kamack first describe that there are 24 permutations of the facelets of a 4-colored piece, comprising the S_4 group (The symmetric group on four letters.) They describe orientations using cycle notation of the four facelets, labeled a, b, c, and d.

The 24 different orientations can be broken down into four crosses, (ab)(cd), (ac)(bd), (ad)(bc), and I, the identity; eight 3-cycles, called twists; six 2-cycles; and six 4-cycles. However, for corner pieces on a cube of any dimension, only even permutations of the facelets can occur, because the odd permutations are mirror images. Thus, 2-cycles and 4-cycles cannot occur because they are odd permutations.

Hence, each corner piece can only be oriented in 12 ways. The even permutations are all possible, and form the alternating group A_4. Keane and Kamack continue by observing that the crosses are a normal subgroup of the alternating group they call N. So,

N = {I, (ab)(cd), (ac)(bd), (ad)(bc)}

The cosets of N form the twists, which they call S and Z:

S = {(abc), (adb), (acd), (bdc)}

Z = {(acb), (abd), (adc), (bcd)}

They then note that the sets N, S, and Z form the quotient-group of A_4 by N, in which N acts as the identity:

$$\frac{A_4}{N} = \{N, S, Z\}$$

We can then see that the group multiplication table is:

	N	S	Z
N	N	S	Z
S	S	Z	N
Z	Z	N	S

From this table, we can see that this quotient-group is isomorphic to the group of residue classes, mod 3. This means that we can assign the number 0 to N, the number 1 to S, and the number -1 to Z, and adding these numbers mod 3 is the same as taking the products of elements of these subgroups.

If we can now show that the sum of the orientations of the corners (counting 0 for an orientation in N, etc.) mod 3 always remains constant, we will be able to determine the final restriction on the number of orientations of the corners. The orientations can be defined by assigning, to each corner, a letter to each facelet and each position of each facelet. Then each orientation can be described by a four-letter string (e.g. ABCD) relative to the position it is occupying.

When pieces rotate in a cycle, their facelets undergo n disjoint cycles if they have n facelets. We can see that every cycle of four corners boils down to four 4-cycles of facelets. We must show that in a 4-cycle of pieces, the sum of the orientations of the pieces mod 3 does not change. The simplest way to do this is to first prove that this is true for a 2-cycle of pieces.

Consider a 2-cycle of corners:

ABCD 1
ABCD 2

Each row represents a corner piece. The 2-cycles of facelets are vertical in direction. For example:

ABCD 1
CDAB 2

This means that facelet A on piece 1 goes where facelet C on piece 2 was, etc. In this example, piece 1 performed an N-twist. Now notice that since we are dealing with cycles, the facelets of piece 2 must return to the original positions of the facelets of piece 1. Therefore, piece 2 also performed an N-twist. It can be checked that if piece 1 performs a Z-twist, piece 2 performs an S-twist, and if piece 1 performs an S-twist, piece 2 performs a Z-twist. Therefore, the sum of the values mod 3 does not change, and equals zero.

Now we simply note that a cycle of pieces of any length can always be expressed as a product of 2-cycles, implying that the sum of the orientation values mod 3 equals zero regardless of the length of the cycles involved. It follows that this is true for the corners of the 2^4 cube as a special case. Since an N-twist is 0, we can have an isolated N-twist without affecting any other pieces. The value S - Z must therefore be congruent to zero, mod 3.

This means that the first 15 corner pieces can each be in any of 12 orientations. If the value of orientations up to that point is 0, the remaining value must be an N-twist. If it is 1, the remaining value must be a Z-twist, and if it is -1, the remaining value must be an S-twist. In each case there are four possible orientations left for the last corner piece. Therefore, the number of orientations the corners can achieve is $12^{16}/3$.

However, we must fix the orientation of the fixed corner as well, making the complete count of configurations of the 2^4 cube:

$$\frac{15!}{2} \cdot \frac{12^{15}}{3} =$$

3357894533384932272635904000.

It should be clear that the arguments presented here apply to the corners of cubes of any size.

4. The 3^4 Cube

Now we will consider the 3^4 cube. We can count that there are 8 immobile centers. There are also 24 2-colored pieces, because each face contains six pieces and each piece lies in two faces, giving $(6\cdot8)/2 = 24$ pieces. Similarly, there are $(12\cdot8)/3 = 32$ 3-colored pieces and of course 16 corner pieces.

Now we once again observe what happens when we rotate a face 90 degrees. We get two 4-cycles of the corner pieces, three 4-cycles of the 3-colored pieces, and one 4-cycle of the 2-colored pieces. We observe that the number of permutations the corners can achieve is once again 16!/2 (we do not need to fix a corner piece this time), and it should be clear that the permutation and orientation counts we found for the corners in the preceding section will hold for cubes of any size, taking into account the fixed corner piece for even cubes.

We also see that the permutations for both the 2-colored and 3-colored pieces are odd, but notice that these odd permutations of pieces must occur together, making their combined parity even. Thus, the number of permutations the 2-colored and 3-colored pieces can achieve is $(24!\cdot32!)/2$. To get the total number of permutations, we multiply these two counts together, obtaining:

$$\frac{24!\cdot32!}{2} \cdot \frac{16!}{2}$$

We must now examine orientations. For the 2-colored pieces, it is clear that in a face rotation there are an even number of 4-cycles of facelets, implying that the orientations of the first 23 pieces determine the orientation of the remaining piece. Therefore, the number of orientations the 2-colored pieces can attain is $2^{24}/2$.

The 3-colored pieces are a bit trickier. Each of them can have 3! = 6 permutations of its facelets. We can see that in a face rotation, there is an odd permutation of the facelets, so it might appear that there are no restrictions on the orientations of the 3-colored pieces. Note, however, that the odd permutation of the facelets occurs along with the odd permutation of the 3-colored pieces themselves, implying that the orientations of the first 31 pieces determine the parity of the orientation of the last piece. This is because the parity of the facelets is determined by the parity of the pieces themselves, and thus the parity of the last piece's orientation is fixed by the parity of the others' orientations, such that their combined parity matches the parity of the permutation of the pieces. Each piece has three odd orientations (three 2-cycles of two of the facelets) and three even orientations (two 3-cycles and the identity). This means that the last piece can only have three different orientations, giving the total number of orientations of the 3-colored pieces as $6^{32}/2$.

The number of orientations of the corner pieces are the same as they were for the 2^4 cube, namely $12^{16}/3$, without the restriction of the fixed corner. Multiplying together the orientation counts for each type of piece, along with the permutation count we obtained above, gives

$$\frac{24!\cdot 32!}{2}\cdot\frac{16!}{2}\cdot\frac{2^{24}}{2}\cdot\frac{6^{32}}{2}\cdot\frac{12^{16}}{3} =$$

17567728807091358431685260790810250596144846301495576514771560217332367989701685506002748876500823542071296000000000000000

as the total number of configurations for the 3^4 cube.

5. A Note on Notation

In order to study cubes larger than the 3^4, it will be useful to develop a better system for naming the pieces under consideration than simply referring to the number of facelets a piece has. This is because on larger cubes, there are often multiple groups of pieces with the same number of facelets.

We shall denote by the term *family* a complete group of pieces that can occupy the same positions on a cube throughout all possible configurations. For example, on a 5^4 cube, there are two families of 3-colored pieces and three families of 2-colored pieces.

The notation system that follows is useful in four dimensions, but is even more valuable in dimensions five and above. The basic idea is to classify families of pieces by the dimension of the section they are located in. To make this more explicit, consider the 4-colored corner pieces. If the Rubik's Cube were replaced by an actual tesseract, a corner would be a point, or dimension zero. So, we will refer to a corner piece as a (0D) piece. 3-colored pieces will be called (1D) pieces, 2-colored pieces will be called (2D) pieces, and 1-colored pieces will be called (3D) pieces. This classification of a piece by the dimension of the section it resides in will be called the *type* of that piece. Also, the term *region* will refer to a section that contains connected pieces of a specific type. For example, on a 5^4 cube, a (1D) region is a 3×1 section that holds three (1D) pieces.

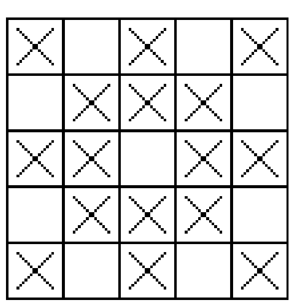
So far, this only appears to be a relabeling of the pieces based on their location on the cube, which seems redundant due to the fact that the number of facelets on a piece is itself determined by the location of that piece on the cube. However, the key idea is this: For (nD) pieces, where $n \geq 3$, we continue to break down the location of the piece until we have $n \leq 2$. For example, consider a 5^4 cube. Its (3D) pieces on a single face form a 3×3×3 cube. The pieces on the corners of that cube will be called (3D)(0D) pieces, the edges, (3D)(1D) pieces, and the faces, (3D)(2D) pieces, while the term (3D) pieces will continue to denote the entire group of pieces. (Note that the term (3D) pieces will often be excluded when referring to types of pieces in general, as it represents multiple piece types.) The center piece will be referred to as a (3D) center, the term center always being reserved for a piece that lies at the center of the region it is located in. It should also be noted that on n^4 cubes, when $n \geq 6$, all of the the (3D) pieces other than the center are broken down into either (3D)(0D), (3D)(1D), or (3D)(0D) pieces regardless of the depth of each piece within the face. For example, take a 7^4 cube. The (3D) pieces in the inner 3×3×3 section will be broken down in the same manner as if they were the (3D) pieces in the 5^4 cube in the example above.

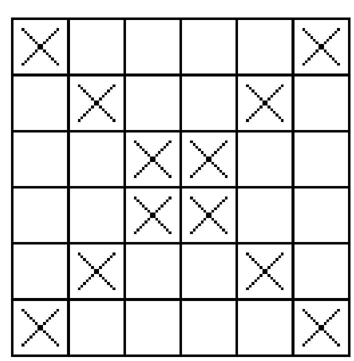
There is another observation to be made regarding the concept of regions. When considering regions for a type of piece that is subclassified (i.e. (3D)(1D) and (3D)(2D) regions for 4-dimensional cubes), a group of pieces of such a type will be broken into separate regions according to the dimension of the subclassification. To make this completely clear, consider a 7^4 cube. A single face has one (3D) region, which is the complete 5×5×5 section of (3D) pieces on that face, as expected. Also, that face has twelve (3D)(2D) regions: Six of them are 3×3 sections of (3D)(2D) pieces (in the outer layer of the (3D) pieces), and the other six are 1×1 sections of (3D)(2D) pieces (in the inner layer of the (3D) pieces). We count these regions as separate, so that each region is a 2-dimensional group of pieces, even though each 3×3 region is connected to the piece of the corresponding 1×1 region. We can also see that there are twelve (3D)(1D) regions, six of which are 3×1 sections of (3D)(1D) pieces, and the remaining six 1×1 sections of (3D)(1D) pieces. It should now be clear what is meant by a (3D)(2D) and (3D)(1D) center on a cube of any size.

This notation system, or one which uses similar concepts, is essential when considering n^d cubes when $d \geq 5$. For example, if $d = 6$, we can have (5D)(4D)(2D) pieces, (4D)(1D) pieces, or even (5D)(4D)(3D)(0D) pieces!

There is one more distinction to be made. We have already defined what we mean by a center piece. The rest are either *wings* or *normals.* Wings are pieces that only occur on a (2D) or (3D)(2D) region. They are the pieces that do not lie on the main diagonals of the (2D) or (3D)(2D) region or (in the case of an odd cube) on the lines that divide the region into four equal square quadrants. Pieces that are neither wings nor centers will be called normals, except for (0D) and (3D)(0D) pieces, which can simply be referred to as pieces without confusion.

To help with visualizing the pieces, here are diagrams of a 5×5 and 6×6 (2D) (or (3D)(2D)) region, in which the normals have been marked with an x:





Finally, here is a list of all possible types of pieces on a 4-dimensional cube:

(0D) pieces (corners)
(1D) centers
(1D) normals
(2D) centers
(2D) normals
(2D) wings
(3D) centers
(3D)(0D) pieces
(3D)(1D) centers
(3D)(1D) normals
(3D)(2D) centers
(3D)(2D) normals
(3D)(2D) wings
(4D) pieces (pieces inside the cube with no facelets)

This system may seem abstract at first, but it will be necessary to understand it well to follow the rest of this paper.

6. The 4^4 Cube

The 4^4 cube will introduce some fundamental concepts which will apply to all larger cubes. We will also begin to study the families of pieces using the notation introduced in the section above.

The following chart and diagram will be used from this point onward to summarize the piece-counting situation:

(0D) pieces - 16
(1D) normals - 64 (64 = (2·12·8)/3)
(2D) normals - 96 (96 = (4·6·8)/2)
(3D)(0D) pieces - 64 (64 = 8·8)

0	1		
	2		

	30		

Each type of piece is represented by a row in this chart. To the right of each hyphen, the number of pieces in each family contained in that type of piece is displayed. (If there is more than one family for a particular type of piece, the number of pieces of each will be listed, separated by a comma.) For each cube chart, calculations will be provided which help explain how the counts were arrived at, presented as they are in this 4^4 chart. The diagram aids in visualizing the families. It displays each layer of a face of the cube from left to right, moving from the outermost layer to the layer nearest the center. (The center layer will not be included for odd cubes.) Each family of pieces will be represented exactly once by displaying the type of that piece, written as 0, 1, 2, 30, 31, or 32, on one of the pieces in the family. Here we see the outer layer of a face on the left, with its (0D), (1D), and (2D) piece families marked appropriately, and the same with the family of (3D)(0D) pieces in the inner layer. We will begin by considering the number of permutations of each family of pieces, starting at the top of the chart.

We first remember that the number of permutations of the (0D) pieces is 15!/2, as always taking the fixed corner into account. Now, consider a 90 degree face rotation. It should be clear that the permutations and orientations of the pieces in each family will be of even parity, and that this is also true for all even cubes. This is due to the fact that the permutations of the pieces and their facelets in each family will consist entirely of an even number of 4-cycles, by the symmetrical nature of the faces of an even cube. We can also make a 90 degree slice rotation, which we can visualize as a normal face rotation in which each type of piece is now located on the corresponding section of the cube of one lower dimension. For the 4^4 cube, the (3D)(0D) pieces are now located in the (2D) pieces' positions, the (2D) pieces are located in the (1D) pieces' positions, and the (1D) pieces are located in the (0D) pieces' positions. The (0D) pieces are never moved by a slice rotation. Additionally, the pieces in the (3D)(0D) pieces' locations are now (4D) pieces, which we do not consider in our analysis because they are inside the cube and are not visible in 4-space. We can see that because a slice move has essentially the same form of a face rotation, with the only modification that the pieces are now in different positions, the permutations of all pieces and their facelets will still be even for all even cubes, and thus we never need to consider odd permutations on such cubes.

The (1D) normals come in 32 pairs of identically colored pieces, so it might at first appear that we must account for this. However, it turns out that two pieces of the same appearance can never occupy the same position and orientation, and are therefore always distinguishable from each other. We can see this by imagining a fourth sticker on each piece in a pair of pieces with the same colors. The sticker on each piece is located on the face of the piece that is adjacent to the other piece (which is hidden from view), so that the two extra stickers touch each other. Notice that this fourth sticker is fixed in place for any position a piece can occupy (always being closest to the center of the (1D) region). Because each piece now has four stickers, and the fourth sticker is fixed in place for each possible position, it follows that an odd permutation of the three real stickers of a piece can never occur for a piece fixed in a certain position. This is because if it were to do so, it would become its own 4-dimensional mirror image.

This implies that each piece can only be in three orientations. (We will get back to this more later.) Now, observe that two matching (1D) normals have the same appearance when viewed in 4-space (when in corresponding orientations), and in order for one to occupy its neighbor's position, it would necessarily flip, thus its facelets would be of opposite parity than its neighbor's would be in the same position. It follows that two identically colored pieces can never occupy the same position and orientation, because if they could, they could be moved to either of the two of their home positions in the same position and orientation, which we just showed is impossible. Thus, the 64 pieces are all distinguishable from each other, and so the number of permutations they can attain is 64!/2.

Another insight will be required to count the number of permutations of the (2D) normals. They come in 24 groups of four matching pieces each, and this time these pieces can occupy the same position and orientation. We see that we should consider identically colored pieces as identical when counting the permutations (so that we never count two positions as separate that differ only in the repositioning of matching pieces amongst themselves). This is because we would be overcounting if we did not do so, when multiplying by the number of orientations. Disregarding parity for now, we can see that the number of permutations is $96!/(24^{24})$. This is because if we consider all of the pieces to be different, we obtain 96! positions. Now notice that each group of four matching pieces can be permuted 4! = 24 different ways amongst themselves, and so there are 24 matching positions for every possible permutation. Therefore, we must divide by 24 for each group of matching pieces, obtaining $96!/(24^{24})$ different positions.

Now, to consider how parity affects our count, we need have but one simple realization - swapping two identical pieces changes the parity, but not the position. This means that parity constraints have no effect when counting the number of permutations of a family with identical pieces. We can see this by observing that we can achieve any of the possible permutations by incorporating a swap of two identical pieces when swapping two others. Therefore, our count above is already correct and does not need to be modified.

Finally, the (3D)(0D) pieces come in eight groups of eight identical pieces each, giving $64!/((8!)^8)$ as the count of these pieces by the reasoning given above. This makes the total number of permutations:

$$\frac{15!}{2} \cdot \frac{64!}{2} \cdot \frac{96!}{24^{24}} \cdot \frac{64!}{(8!)^8}$$

Now for orientations. We already know that the orientation count for the corners is $12^{15}/3$. For the (1D) normals, we will have to make one more important observation. We already know that there are three possible orientations for each piece. The only thing left to consider is how the orientations of the first 63 affect the orientation of the last one. We can use an argument similar to Keane and Kamack's reasoning above for the orientations of the corners to show that the last piece is fixed by the first 63. To see this, we have as the possible permutations of facelets the alternating group A_3. This time, we have the identity, I, the 3-cycle (abc) which we will call an S-twist, and the 3-cycle (acb) which we will call a Z-twist. The multiplication table is once again:

	I	S	Z
I	I	S	Z
S	S	Z	I
Z	Z	I	S

This implies that we can count I as 0, S as 1, and Z as -1, and adding these numbers mod 3 is identical to performing each twist, as before. The same reasoning as before also shows that the sum of these orientation values mod 3 is always zero. Therefore, the sum of the orientation values of the first 63 (1D) normals determines the twist of the last, and the count for the number of orientations of the (1D) normals is $3^{64}/3$.

The (2D) normals are simple, as we know that the parity of the (2D) normals' facelets is even, thus implying that the orientation count is $2^{96}/2$. Finally, the (3D)(0D) pieces have no orientation, thus making the complete count for the number of configurations of the 4^4 cube:

$$\frac{15!}{2} \cdot \frac{64!}{2} \cdot \frac{96!}{24^{24}} \cdot \frac{64!}{(8!)^8} \cdot \frac{12^{15}}{3} \cdot \frac{3^{64}}{3} \cdot \frac{2^{96}}{2} =$$

130465639524605309368634620044528122859025488438611959323482221544701493566589669139
598204956926940147059366252849247482898636104705417194760866897307590845202461293100
468293214262958591194739437727430945469384490361714647847550801897750293894453665815
5728292577589074251289198088626162596049972101120000000000000000000000000000000000000

000000000.

It can be seen that the reasoning for the permutations and orientations of the (1D) normals apply to the (1D) normals of a cube of any size, with the only change in our imaginary stickers argument being to place the imaginary sticker on the face of the (1D) normal that is closest to the center of the (1D) region, as the (1D) normals on larger cubes are not always adjacent to the other (1D) normal in their family. Also, the reasoning for families with identical pieces clearly holds for any such group.

7. The 5^4 Cube

Here are the piece-counting chart and diagram for the 5^4 cube:

(0D) pieces - 16
(1D) centers - 32 ($32 = (12\cdot 8)/3$)
(1D) normals - 64 ($64 = (2\cdot 12\cdot 8)/3$)
(2D) centers - 24 ($24 = (6\cdot 8)/2$)
(2D) normals - 96, 96 ($96 = (4\cdot 6\cdot 8)/2$)
(3D)(0D) pieces - 64 ($64 = 8\cdot 8$)
(3D)(1D) centers - 96 ($96 = 12\cdot 8$)
(3D)(2D) centers - 48 ($48 = 6\cdot 8$)

0	1	1		
	2	2		
		2		

	30	31		
		32		

We can see that the number of types and families of pieces is growing quickly as we increase the size of the cube. We have our 16 (0D) pieces as always. The (1D) pieces reside in a group of 3×1 (1D) regions, with the counts given above. The (2D) pieces are located in a group of 3×3 (2D) regions, in which there are two families of (2D) normals, and one of (2D) centers. Finally, the (3D) pieces are divided into three separate subtypes: The (3D)(0D) pieces, the (3D)(1D) centers, and the (3D)(2D) centers. Of course, we do not consider the (3D) centers as they are our immovable reference points.

In a 90 degree face rotation, we find odd permutations of the (1D) centers, (2D) centers, (3D)(1D) centers, and (3D)(2D) centers, and even permutations for all other families of pieces. We see that we do not need to consider that the (3D)(1D) and (3D)(2D) centers come in odd permutations, because of the fact we discovered earlier that parity constraints do not apply to families that contain identical pieces. Therefore, we only need to consider odd permutations of the (1D) and (2D) centers. With a bit of thought, it can be seen that this is true for odd cubes of any size, as all of the other families of pieces either contain identical pieces, or are (1D) normals, which always come in even permutations. If we now consider the slice moves of an odd cube of any size (realizing that there is more than one slice move for cubes of size 6^4 and larger), we see that the only families of pieces that have odd permutations are (2D) normals and (3D) pieces. (2D) normals and (3D) pieces always come in families that contain identical pieces, so they do not need to be considered with regards to parity.

Thus, we have proven that on an odd cube of any size, the only families that need to be addressed for having odd permutations of their pieces are (1D) and (2D) centers. All other families either come in even permutations, or contain identical pieces and do not need to be considered.

As for orientations, on any odd cube we see that only (1D) centers need to be addressed for having an odd permutation of facelets. This is because we have already found (0D) pieces and (1D) normals to come in even orientations, (2D) pieces contain two facelets per piece, and thus come in even permutations of facelets, and (3D) pieces have no orientations. Slice moves never move (1D) centers, and hence we have proven that on an odd cube of any size, the family of (1D) centers always has an odd permutation of its facelets, and is the only family of pieces that does so.

This analysis of the parity situation of odd cubes of any size, along with our similar study of even cubes of any size, will make things much easier in the sections to come. We will now begin counting the permutations of the 5^4 cube, starting at the top of the chart.

We already know that the corners have 16!/2 permutations. By following the same reasoning we used for the 3^4 cube, we see that the number of permutations of the (1D) and (2D) centers is (24!·32!)/2. For the (1D) normals we remember, as we mentioned in the previous section on the 4^4 cube, that the count obtained there will apply to the (1D) normals on any cube. Thus, the number of permutations of the (1D) normals is 64!/2. Each of the two families of (2D) normals come in 24 groups of 4 identical pieces each, and so the count for each family is $96!/(24^{24})$. Similarly, the count for the (3D)(0D) pieces is $64!/((8!)^8)$, the count for the (3D)(1D) centers is $96!/((12!)^8)$, and the count for the (3D)(2D) centers is $48!/((6!)^8)$. Multiplying all of these counts together, we obtain

$$\frac{16!}{2}\cdot\frac{24!\cdot 32!}{2}\cdot\frac{64!}{2}\cdot\left(\frac{96!}{24^{24}}\right)^2\cdot\frac{64!}{(8!)^8}\cdot\frac{96!}{(12!)^8}\cdot\frac{48!}{(6!)^8}$$

as the number of permutations of the 5^4 cube.

We will now study orientations. We know that the corners have $12^{16}/3$ orientations, as before. The (1D) centers have $6^{32}/2$ orientations, by the same logic that was used in the 3^4 section. Additionally, the (1D) normals have $3^{64}/3$ orientations, because the reasoning in the previous section for the (1D) normals on the 4^4 cube applies to all larger cubes. The (2D) centers have $2^{24}/2$ orientations, and each family of (2D) normals has $2^{96}/2$ orientations, as we proved above they come in even permutations of facelets. Of course, the (3D) pieces have no orientations. Multiplying all of our counts together, we obtain as the number of configurations of the 5^4 cube:

$$\frac{16!}{2}\cdot\frac{24!\cdot 32!}{2}\cdot\frac{64!}{2}\cdot\left(\frac{96!}{24^{24}}\right)^2\cdot\frac{64!}{(8!)^8}\cdot\frac{96!}{(12!)^8}\cdot\frac{48!}{(6!)^8}\cdot\frac{12^{16}}{3}\cdot\frac{6^{32}}{2}\cdot\frac{3^{64}}{3}\cdot\frac{2^{24}}{2}\cdot\left(\frac{2^{96}}{2}\right)^2=$$

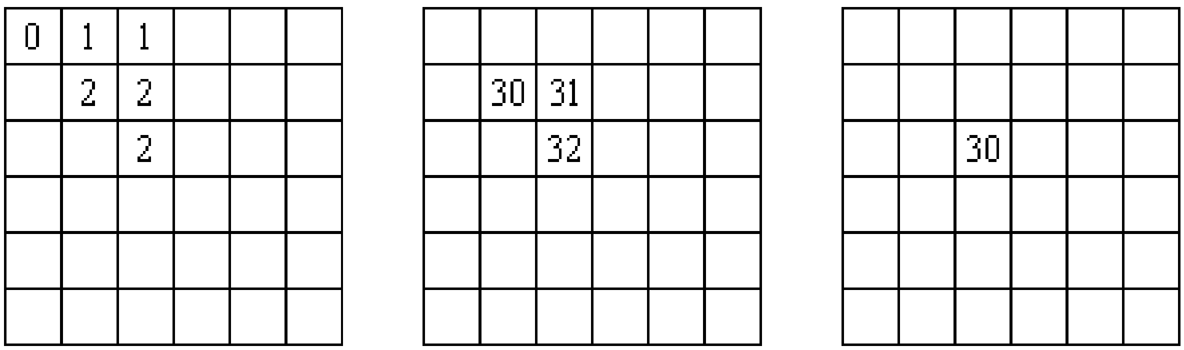
123657056923899002698227805778387808933769666084597331170345244675638825481620700008
237306084142730598637705860008300844182287747674018136874315751080178664887107264876
848935590538625767958284656419396560246923935065962447405384165866873326263467921778
683862961389770831926039889601733193275112578283448018613526925847925558456540351327
099176534335451141045209002537535755031468961150691008214712492137716092251416854303
972448469954444917129644451683375275906483623456408625743663232956462751569098735992
247230927473597130714467427915529825001467413803400014037257220682520596555932663885

3240055395996672769449263104000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000.

8. The 6^4 Cube

We begin our analysis of the 6^4 cube with its piece-counting chart and diagram:

(0D) pieces - 16
(1D) normals - 64, 64 (64 = (2·12·8)/3)
(2D) normals - 96, 96 (96 = (4·6·8)/2)
(2D) wings - 192 (192 = (8·6·8)/2)
(3D)(0D) pieces - 64, 64 (64 = 8·8)
(3D)(1D) normals - 192 (192 = 2·12·8)
(3D)(2D) normals - 192 (192 = 4·6·8)



We see that a (1D) region contains two families of (1D) normals, each with 64 pieces. The diagram and calculations above should also make the counts for the (3D) pieces clear, noting that a (3D)(1D) normal can be brought to its neighbor's position in its (3D)(1D) region with a minimum of three 90 degree face rotations. It should begin to be apparent that in any type of piece, each family will contain a constant number of pieces, regardless of the size of the cube or the number of families that type of piece contains.

In a (2D) region, there are two families of (2D) normals and one of (2D) wings. It is not immediately clear whether there should be one or two families of (2D) wings; at first glance it may not appear that such a piece can be moved to occupy its neighbor's position. In fact, this cannot be done by rotating a single face, but it can be accomplished by rotating multiple faces. The simplest way to do this would be to rotate three adjacent faces so that the outer layer that the piece is located in is first removed from its face, then brought back to its original position flipped.

It can now be determined that the orientation of each (2D) wing is fixed for each position it can occupy. This implies that a (2D) wing and its neighbor can never be in the same position and orientation. Our method will be similar to the method used to establish the analogous result for the (1D) normals.

We imagine attaching a third sticker to the face of an arbitrary (2D) wing that is touching its neighbor (this sticker is hidden from view in 4-space), and observe that this sticker is fixed in place for each position the piece can occupy, always being next to its neighbor. Next, we attach a fourth sticker to the face of the (2D) wing that is adjacent to the (2D) normal in the innermost layer of the face, and again note that this sticker is fixed in place for any position the piece can occupy. Hence, we can think of each (2D) wing as a 4-colored piece.

Therefore, because two of the four stickers of a (2D) wing are fixed for any position the piece can occupy, the other two must be as well, for otherwise the piece would become its own mirror image, which is not possible. This implies that a (2D) wing and its neighbor can never be in the same position and orientation, as for one to occupy its neighbor's position it would necessarily flip, causing its real stickers to be of opposite parity than its neighbor's would be in that position. It can be observed that these results for the (2D) wings apply to the (2D) wings of a cube of any size, the only change in the arguments being to place the additional stickers on the faces of each piece which correspond to the faces with additional stickers on the 6^4 cube. These two stickers are always uniquely identified because they are always different distances from the edge of the cube.

We are now ready to count the permutations and orientations of each family of pieces in the 6^4 cube. We will start with the permutations, at the top of the chart.

The corners, of course, have 15!/2 permutations. We know from previous sections that each family of (1D) normals has 64!/2 permutations, giving $(64!/2)^2$ as the count for both families. We also know that the (2D) normals have $96!/(24^{24})$ permutations for each family, making the total $(96!/(24^{24}))^2$ for both. The (2D) wings are a family of 192 pieces, but there are identical pieces in this group. Because this family comes in groups of eight identically colored pieces, and a piece and its neighbor can never be in the same position and orientation, it follows that the (2D) wings consist of 48 groups of 4 identical pieces each. Thus, there are $192!/((4!)^{48}) = 192!/(24^{48})$ permutations of the (2D) wings. The (3D) pieces follow the same rules for families with identical pieces: There are $(64!/((8!)^8))^2$ permutations of the (3D)(0D) pieces, $192!/((24!)^8)$ permutations of the (3D)(1D) normals, and $192!/((24!)^8)$ permutations of the (3D)(2D) normals. Multiplying these permutation counts together, we obtain the number of permutations of the 6^4 cube:

$$\frac{15!}{2}\cdot\left(\frac{64!}{2}\right)^2\cdot\left(\frac{96!}{24^{24}}\right)^2\cdot\frac{192!}{24^{48}}\cdot\left(\frac{64!}{(8!)^8}\right)^2\cdot\left(\frac{192!}{(24!)^8}\right)^2$$

Now we will count the orientations. The corners have $12^{15}/3$ orientations. We know that the (1D) normals have $3^{64}/3$ orientations for each family, making the total $(3^{64}/3)^2$ for both. Each family of (2D) normals has $2^{96}/2$ orientations, giving the total for both families as $(2^{96}/2)^2$. We discovered that the (2D) wings have no orientations. Finally, the (3D) pieces also have no orientations. Multiplying the permutation counts above with the ones for orientations, we find the total number of configurations of the 6^4 cube to be:

$$\frac{15!}{2}\cdot\left(\frac{64!}{2}\right)^2\cdot\left(\frac{96!}{24^{24}}\right)^2\cdot\frac{192!}{24^{48}}\cdot\left(\frac{64!}{(8!)^8}\right)^2\cdot\left(\frac{192!}{(24!)^8}\right)^2\cdot\frac{12^{15}}{3}\cdot\left(\frac{3^{64}}{3}\right)^2\cdot\left(\frac{2^{96}}{2}\right)^2 =$$

2643432397631320778500134553673958820699207649151766176158964256047726173954767918O7
544912068783367475497344654390039776935146828007877209739947496200882251028332070620
913612639733391972191751218779811162066518418201513821485710066286540019140424063030
1429360363214996466712438873660801491292308642492149535607273106085350108782380671O5
1963271523544294328364145248427890776457184978640654950847770428421062088140238896 36
2236296493402584602040115732610466094292728150622657511115176061113863362557029040 31
7614689746950358557206743419430752323016151867802448776276366566628808472719092666 95
1780665515736532736561912782744002646291923277900873397568402445953724930681609333 47
4034605162499195128015278995981839850617191981306617598468452192629812680147093400 65
0536820032857040975954917719537114553138767596948755609168286604542774467832409052 33
4187639990066505476689708752370694768015380629638798967171363810339619450313663949 41

725708248736390551997180317157379215039227670778812154285466911957373591754065087207
314000103891688829357492770928907438925806912248892452824237313989962030484325621500
26881388301680805348955557724160000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000.

9. The 7^4 Cube

The 7^4 Cube will be simpler than the preceding sections, as no new deductions will be needed to find the number of configurations. Here are its piece-counting chart and diagram:

(0D) pieces - 16
(1D) centers - 32 $(32 = (12\cdot 8)/3)$
(1D) normals - 64, 64 $(64 = (2\cdot 12\cdot 8)/3)$
(2D) centers - 24 $(24 = (6\cdot 8)/2)$
(2D) normals - 96, 96, 96, 96 $(96 = (4\cdot 6\cdot 8)/2)$
(2D) wings - 192 $(192 = (8\cdot 6\cdot 8)/2)$
(3D)(0D) pieces - 64, 64 $(64 = 8\cdot 8)$
(3D)(1D) centers - 96, 96 $(96 = 12\cdot 8)$
(3D)(1D) normals - 192 $(192 = 2\cdot 12\cdot 8)$
(3D)(2D) centers - 48, 48 $(48 = 6\cdot 8)$
(3D)(2D) normals - 192, 192 $(192 = 4\cdot 6\cdot 8)$

0	1	1	1			
	2	2	2			
		2	2			
			2			

	30	31	31			
		32	32			
			32			

		30	31			
			32			

As can be seen, all piece types are represented in this cube except for (3D)(2D) wings. We will begin by counting the permutations.

There are $16!/2$ corner permutations. We remember from previous sections that on an odd cube, the number of permutations of the (1D) and (2D) centers is $(24!\cdot 32!)/2$. There are two families of (1D) normals, making the count for both $(64!/2)^2$. The (2D) normals come in four families, resulting in $(96!/(24^{24}))^4$ permutations. There is one family of (2D) wings, as in the last section, and the count is once again $192!/(24^{48})$. Counting the permutations of the (3D) pieces is straightforward, as long as we make sure to count the number of families in each type of piece properly. There are $(64!/((8!)^8))^2$ permutations of the (3D)(0D) pieces, $(96!/((12!)^8))^2$ permutations of the (3D)(1D) centers, $192!/((24!)^8)$ permutations of the (3D)(1D) normals, $(48!/((6!)^8))^2$ permutations of the (3D)(2D) centers, and

$(192!/((24!)^8))^2$ permutations of the (3D)(2D) normals. We obtain the total number of permutations by multiplying all of these counts together:

$$\frac{16!}{2}\cdot\frac{24!\cdot 32!}{2}\cdot\left(\frac{64!}{2}\right)^2\cdot\left(\frac{96!}{24^{24}}\right)^4\cdot\frac{192!}{24^{48}}\cdot\left(\frac{64!}{(8!)^8}\right)^2\cdot\left(\frac{96!}{(12!)^8}\right)^2\cdot\left(\frac{48!}{(6!)^8}\right)^2\cdot\left(\frac{192!}{(24!)^8}\right)^3$$

We will now count the number of orientations. The corners have $12^{16}/3$ orientations, as always on odd cubes. We recall from before that there are $6^{32}/2$ orientations of the (1D) centers. For the (1D) normals, we count $(3^{64}/3)^2$ orientations, remembering to count once for each family. There are $2^{24}/2$ orientations of the (2D) centers, as before. Finally, we have $(2^{96}/2)^4$ orientations of the (2D) normals; there are no orientations for any of the remaining families. Multiplying everything together, we obtain the number of configurations of the 7^4 cube:

$$\frac{16!}{2}\cdot\frac{24!\cdot 32!}{2}\cdot\left(\frac{64!}{2}\right)^2\cdot\left(\frac{96!}{24^{24}}\right)^4\cdot\frac{192!}{24^{48}}\cdot\left(\frac{64!}{(8!)^8}\right)^2\cdot\left(\frac{96!}{(12!)^8}\right)^2\cdot\left(\frac{48!}{(6!)^8}\right)^2\cdot\left(\frac{192!}{(24!)^8}\right)^3\cdot$$
$$\frac{12^{16}}{3}\cdot\frac{6^{32}}{2}\cdot\left(\frac{3^{64}}{3}\right)^2\cdot\frac{2^{24}}{2}\cdot\left(\frac{2^{96}}{2}\right)^4=$$

733743431989203499653969654101590141517645746039252846362558145764019036511682339053
046802371562652660442960696980561660162897005188088822113491373316524207715498428153
089868921026967994146075904281768384493308985145369878686479450986334974197030255160
202722503934784368170544665725854546173956681390863133658159042053262508329517666310
178084117766493933109622945245176134150971217934827165414663523220620725714521754301
820725680690311197994116614091110218043224578431745458391890473938459448319762318337
664299733533447880542620950263954589748078364787091625469688291726407353272805727692
923868712100367788243482643376813708488356094288175471398841113769565782775558122047
534189235070031586358401932011679947427194177064043049709192489364793276911138702316
449614036570516207352280544798143723706079732591151233363224532429457109482886115394
814664242106749491856028058426358397493326266018892320583091614729413155005749797571
359784100582075686014254255227213647353814393502791946516994430276229498052371986224
617477487398563652861387582456733327424716666006513626378064106148971295020871194488
017655844355526081653094523231897759871825388018810225231095005716852714319343434690
215559790534984700328221541796279063270248668545434765890862906873626153945483927658
821257201550955756583206864440214742450719080680231840149496629020896736673985073830
598202620736351606098826255055851007156367599417271409095955425254654973644440441852
829766581221333799477282417693119951892365111281198919248889233138780723461052256343
277296703684670010092638255885840093075248166344842794314031222291602005573986495784
245004165291612893751320471626039527879045748264679735739118512596870138536907514962
293170143410488629222126623896234205841145138109204624801344885275316474038318367057
373475691723100401968708263166415392143434444379750327134453760000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000.

10. The 8^4 Cube

By this point, it should be evident that for each type of piece, we only need to calculate the number of permutations and orientations for a family once. These counts will hold for all larger cubes, and all we need to do is find the number of families for a particular cube. When we find the general formula, we will use our previously calculated permutation and orientation counts for each type of piece; the main work will be finding the number of families for each type on an arbitrarily sized cube. Then, to count the number of permutations and orientations for each type of piece, we apply our term which counts the number of families for that type as an exponent to the permutation and orientation counts for that type.

Here are the piece-counting chart and diagram for our last cube:

(0D) pieces - 16
(1D) normals - 64, 64, 64 ($64 = (2\cdot12\cdot8)/3$)
(2D) normals - 96, 96, 96 ($96 = (4\cdot6\cdot8)/2$)
(2D) wings - 192, 192, 192 ($192 = (8\cdot6\cdot8)/2$)
(3D)(0D) pieces - 64, 64, 64 ($64 = 8\cdot8$)
(3D)(1D) normals - 192, 192, 192 ($192 = 2\cdot12\cdot8$)
(3D)(2D) normals - 192, 192, 192 ($192 = 4\cdot6\cdot8$)
(3D)(2D) wings - 192, 192 ($192 = 4\cdot6\cdot8$)

0	1	1	1				
	2	2	2				
		2	2				
			2				

	30	31	31				
		32	32	32			
			32				

		30	31				
			32				

			30				

The only additional explanation required in this section deals with the (3D)(2D) wings, the one type of piece that remains unexamined. We must show that this piece can never occupy its neighbor's position, and thus has the same permutation count as the (3D)(2D) normals, namely $192!/((24!)^8)$. To start, notice that as a (3D) piece, a face rotation can only move a (3D)(2D) wing 3-dimensionally, since it lies on the face itself. Such a rotation cannot bring it to its neighbor's position. The last possibility to consider is that of a slice move. To show that a slice move cannot bring a (3D)(2D) wing to its neighbor's position, it will be easier just to show that it can never move to that position, by means of our imaginary stickers argument. Imagine three additional stickers: One is located adjacent to the (3D)(2D) normal nearest to the center of the (3D)(2D) region, one is located adjacent to its neighbor, and one is located adjacent to the (3D)(1D) normal in the third layer. Clearly, no slice rotation can reposition these three stickers for any particular piece. Also, the real sticker must remain in place, on the face of the cube. Therefore, the piece is completely fixed in place for each position it can occupy. In order to occupy its neighbor's position, the parity of the faces of the piece would have to change. Since it cannot, the piece must remain in its original position when in that pair of pieces. These arguments also hold for the (3D)(2D) wings in cubes of any size, as we can place the additional stickers on the corresponding faces of the (3D)(2D) wing. These faces can never be confused for each other because they are always different distances from the edge of the cube.

Now we can count the number of permutations of our last cube. There are $15!/2$ permutations of the corner pieces, $(64!/2)^3$ permutations of the (1D) normals, $(96!/(24^{24}))^3$ permutations of the (2D) normals, $(192!/(24^{48}))^3$ permutations of the (2D) wings, $(64!/((8!)^8))^3$ permutations of the (3D)(0D) pieces, $(192!/((24!)^8))^3$ permutations of the (3D)(1D) normals, $(192!/((24!)^8))^3$ permutations of the (3D)(2D) normals, and $(192!/((24!)^8))^2$ permutations of the (3D)(2D) wings. Multiplying these gives

$$\frac{15!}{2}\cdot\left(\frac{64!}{2}\right)^3\cdot\left(\frac{96!}{24^{24}}\right)^3\cdot\left(\frac{192!}{24^{48}}\right)^3\left(\frac{64!}{(8!)^8}\right)^3\cdot\left(\frac{192!}{(24!)^8}\right)^7$$

as the number of permutations for the 8^4 cube.

Now we will count orientations. There are $12^{15}/3$ orientations of the corner pieces, $(3^{64}/3)^3$ orientations of the (1D) normals, and $(2^{96}/2)^3$ orientations of the (2D) normals. Multiplying the permutation and orientation counts together, we obtain the number of configurations of the 8^4 cube to be:

$$\frac{15!}{2}\cdot\left(\frac{64!}{2}\right)^3\cdot\left(\frac{96!}{24^{24}}\right)^3\cdot\left(\frac{192!}{24^{48}}\right)^3\cdot\left(\frac{64!}{(8!)^8}\right)^3\cdot\left(\frac{192!}{(24!)^8}\right)^7\cdot\frac{12^{15}}{3}\cdot\left(\frac{3^{64}}{3}\right)^3\cdot\left(\frac{2^{96}}{2}\right)^3=$$

4515047282974184492907566879672990236570536312765072951113978309962371480970756245847
6234096537982372322616530195956647145074456250824723485998924862223150260537773084454
4845297155272655046044113382851451203758957765590527184505890036599828439273137505755
0485445929099109155753654453879073591081165678123741479467375277546193877295561535711
2295929282469299957146303970388930456885725610053815828489022074574030156994557881988
7823822806547209120583502844382587108317268098671438493108160238421286226717681330933
6196582476799802940207273030119790665543909939173391418806322499216962736187890712233
6503256951669387535109798010129927373554333948447123730416954286639504039686462484666
1545286908450610789504355625533936944501631387895267142527632876521658903961635153355
8406274942846764972464716561834013772652482776921089219061184695247097169026260178188
5545277186715437550827239353757222466515499176244157312716007136397552213702090614233
6687046909689401703643464613287104439791401858851178735840126409899767876969032199899
452495453893436665465278980770490150717336066122857362521651097667352231645631896109

952987911206614699004715285835379340456632728067903563331406019943538833063370285603616020379414986374060688390892952042498237107281869741512453588585458536588024010131292866599997562622913942975213348439579578416613825438772356403707541718110400255267061951661405192634981206342893316593846462401491070461583199357126729859678349805529886769345460775617838343278353216625144784071688992620544384325029553329513793185147148855906221950998170267291475500277329608683319879521221290824956930911758156640655705541364373093193095575133020417847304491102574627634986498638790009594516697743827740639375999751303342871332214227945546757605526046505313873857720055308129788279353106927167296055309382018459041396204211193054246894619908654998300105761177522636420523716164811115909267215676523249092572740108618793474896418482132670980271085479341868961206385772288953728732074380358883591952505112085048769606360682730384011435609545205494362248142873782712377192611203853650070950824703495500979700023338656129453162485665359635540368645842725176332666444575551809717568328660554942049301641540024221819140435281858830124500922747553517135673188514475174987136027464627734946649911438418896999801622766113193616344166665722211046216771114914810202389221454307480373144530176243238472311777886226645282531115666004473545890791851545283773968906877609531644748495910146478776074353015282746772072456903818936353099714387027464622321369357804826248043834052140479749863025238725325729869405598009102220809003559614117027187790879558820152485725393879324974435532800.

11. The Generalization

After examining all of the previous cubes and their piece types, we are ready to generalize our findings to a cube of any size. The procedure we will use will be to examine each type of piece one at a time. For each piece type, we will list our previously derived number of permutations and orientations for a family of that piece. Then, we will deduce the number of families for that type of piece on an n^4 cube. This expression will be applied as an exponent to our calculation for the permutation and orientation counts for a single family, which results the number of configurations of all pieces of that type on a cube of any size. When we multiply these terms for each type of piece together, we will obtain $C_4(n)$, the number of configurations of an n×n×n×n Rubik's Cube. We will begin with the corners, (1D) centers, and (2D) centers.

Step 1: The (0D) pieces, (1D) centers, and (2D) centers

We will start with the corners. We already know the number of configurations of the corners for an arbitrarily sized cube: For an even cube, it is $(15!/2)(12^{15}/3)$, and for an odd cube it is $(16!/2)(12^{16}/3)$. We will rewrite these two calculations as a single term using the modulo operation: (n mod 2) is 0 when n is even, and 1 when n is odd. Therefore, we can see that

$$\frac{15!\cdot 12^{15}}{6}(16\cdot 12)^{n \bmod 2}$$

correctly counts the number of configurations of the corners for an n^4 cube.

Now we will consider (1D) and (2D) centers. These only occur on an odd cube, so we will once again use the (n mod 2) term as an exponent. Clearly, we only have one family each of (1D) and (2D) centers for a cube of any size. The number of permutations of the (1D) and (2D) centers together is (24!·32!)/2. Also, the number of orientations of the (1D) centers is $6^{32}/2$, and the number of orientations of the (2D) centers is $2^{24}/2$, or 2^{23}. Combining these and simplifying, we obtain the number of configurations of the (1D) and (2D) centers:

$$\left(24!\cdot 32!\cdot 2^{21}\cdot 6^{32}\right)^{n \,\mathbf{mod}\, 2}$$

Note that this equals one when n is even, and thus will be cancelled out when multiplied by the other terms in our formula. We will now combine the two terms together. We can rewrite 16·12 from the corner calculation as $2^5\cdot 6$, and multiply this by our (1D) and (2D) center count to obtain

$$\frac{15!\cdot 12^{15}}{6}\left(24!\cdot 32!\cdot 2^{26}\cdot 6^{33}\right)^{n \,\mathbf{mod}\, 2}$$

as the number of configurations of the corners, (1D) centers, and (2D) centers.

Step 2: The (1D) normals

Next we will consider the (1D) normals. We know from before that the number of permutations and orientations of a family of (1D) normals is $(64!/2)\cdot 3^{63}$, and that this number holds for a cube of any size. We now need to count the number of families of (1D) normals on an n^4 cube.

(1D) normals only exist on cubes of size 4^4 and larger. Visualizing a (1D) region, we see that each piece other than the center on an odd cube is a (1D) normal. Furthermore, each pair of (1D) normals equidistant from the center belong to one family. On an n^4 cube, a (1D) region is an (n - 2)×1 array of pieces. Therefore, on an even cube, the number of families of (1D) normals is (n - 2)/2, and on an odd cube it is (n - 3)/2. We can write these two counts as one expression using the floor function, which is equal to the largest integer less than or equal to the number it is affecting. Using the floor function, we can write the number of families of (1D) normals on an n^4 cube as $\lfloor (n-2)/2 \rfloor$. This results in our calculation for the number of configurations of the (1D) normals to be:

$$\left(\frac{64!}{2}\cdot 3^{63}\right)^{\left\lfloor \frac{n-2}{2} \right\rfloor}$$

Note that this term equals 1 when n equals 2 or 3, which correctly counts no (1D) normals when multiplied by the other terms in the formula.

Step 3: The (2D) normals

Now for the (2D) normals. We have previously counted the number of permutations and orientations of a family of these pieces to be $(96!/(24^{24}))\cdot 2^{95}$. We will now count the number of families of (2D) normals on a cube of any size.

The (2D) normals lie within an (n - 2)×(n - 2) (2D) region on an n^4 cube with $n \geq 4$. On an even cube, they lie on the diagonals. On an odd cube, they lie both on the diagonals and what we will call the straights - (2D) normals or (3D)(2D) normals that are not diagonals. We can see that the number of families of diagonal pieces in an even cube is half of the number of layers of the (2D) region, as there is one family of (2D) normals for each layer above the center. This gives (n - 2)/2 as the number of families of the (2D) normals on an even cube. On an odd cube, the number of families of the diagonals will be (n - 3)/2, because we must not count the center layer of the (2D) region. We can see that the number of families of straights will also be (n - 3)/2, by the same reasoning. We can write the number of families of the diagonals in either an even or odd cube as $\lfloor (n-2)/2 \rfloor$, and then use the (n mod 2) term to add an additional (n - 3)/2 if n is odd. Doing this, we obtain

$$\left(\frac{96!}{24^{24}}\cdot 2^{95}\right)^{\left\lfloor \frac{n-2}{2} \right\rfloor + (n \,\mathbf{mod}\, 2)\left(\frac{n-3}{2}\right)}$$

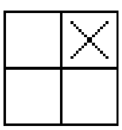
as the number of configurations of (2D) normals. This term equals 1 when n is 2 or 3, as desired.

Step 4: The (2D) wings

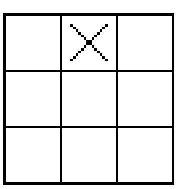
We will now study the (2D) wings. We recall that the number of permutations and orientations of a family of (2D) wings is $192!/(24^{48})$. Now we will count the number of families.

There are no (2D) wings when n < 6 on an n^4 cube. Let us take a look at some diagrams, which display one quadrant of an (n - 2)×(n - 2) region, including the center layer for odd cubes, starting with n = 6. We only need to look at a quadrant because each quadrant contains the same pieces and families, and we only need to count one piece per family. Here are the diagrams, with the pieces representing each family of (2D) wings marked with an x:

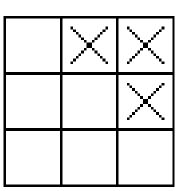
n = 6:



n = 7:



n = 8:



n = 9:

	X	X	
		X	

n = 10:

	X	X	X
		X	X
			X

n = 11:

	X	X	X	
		X	X	
			X	

When n is equal to 6 or 7, we have one family, when n is equal to 8 or 9, we add two to one to get three families, and when n is equal to 10 or 11 we add three to three to get six families. It should be clear that this pattern continues; we have encountered triangular numbers, defined by the sequence 1, 1 + 2 = 3, 1 + 2 + 3 = 6, 1 + 2 + 3 + 4 = 10, etc. It is known that the formula for the mth triangular number is m(m + 1)/2, all we must do now is correctly represent m as an expression in n to obtain the number of families. Here is a table that identifies the correct value of m for each value of n, for $2 \leq n \leq 13$:

n = 2: m = 0
n = 3: m = 0
n = 4: m = 0
n = 5: m = 0
n = 6: m = 1
n = 7: m = 1
n = 8: m = 2
n = 9: m = 2
n = 10: m = 3
n = 11: m = 3
n = 12: m = 4
n = 13: m = 4

We can see that the formula for m is m = $\lfloor (n-4)/2 \rfloor$ when n ≥ 4. For the moment we do not need to worry that this does not hold when n is 2 or 3. Plugging this value of m into the triangular number formula, we obtain the number of families of (2D) wings to be $\lfloor (n-4)/2 \rfloor (\lfloor (n-4)/2 \rfloor + 1)/2$, which equals $\lfloor (n-4)/2 \rfloor \lfloor (n-2)/2 \rfloor /2$. Note that this formula is correct for n equal to 2 or 3. Therefore, the number of configurations of the (2D) wings is:

$$\left(\frac{192!}{24^{48}} \right)^{\frac{\left\lfloor \frac{n-4}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor}{2}}$$

Step 5: The (3D)(0D) pieces

To begin with the (3D)(0D) pieces, we remember that the number of permutations of a family is $64!/((8!)^8)$. They lie within an (n - 2)×(n - 2)×(n - 2) (3D) region, and we see that there is one family for each layer in the region above the center. This makes the number of families (n - 2)/2 for even cubes and (n - 3)/2 for odd cubes, which we can write generally as $\lfloor (n-2)/2 \rfloor$. This results in

$$\left(\frac{64!}{(8!)^8} \right)^{\left\lfloor \frac{n-2}{2} \right\rfloor}$$

as the number of configurations of the (3D)(0D) pieces.

Step 6: The (3D)(1D) centers

The (3D)(1D) centers have $96!/((12!)^8)$ permutations for each family. They only exist on odd cubes, in an (n - 2)×(n - 2)×(n - 2) (3D) region. As with the (3D)(0D) pieces, each layer above the center layer contains one family, making the number of families (n - 3)/2. Applying the modulo operation, we find the number of configurations of the (3D)(1D) centers to be:

$$\left(\frac{96!}{(12!)^8} \right)^{(n \bmod 2)\left(\frac{n-3}{2} \right)}$$

Step 7: The (3D)(2D) centers

The (3D)(2D) centers have $48!/((6!)^8)$ permutations per family. The number of families is identical to to the (3D)(1D) centers; they come on odd cubes with each layer of the (n - 2)×(n - 2)×(n - 2) region above the center layer having one family. Therefore, the number of configurations of the (3D)(2D) centers is:

$$\left(\frac{48!}{(6!)^8} \right)^{(n \bmod 2)\left(\frac{n-3}{2} \right)}$$

Step 8: The (3D)(1D) normals, (3D)(2D) normals, and (3D)(2D) wings

Each family of the (3D)(1D) normals, (3D)(2D) normals, and (3D)(2D) wings contains the same number of permutations and orientations, namely $192!/((24!)^8)$. Therefore, we can count the number of families of each of them, and add these to obtain the total number of families. Let us begin with the (3D)(1D) normals.

(3D)(1D) normals only exist on cubes where $n \geq 6$, and occur in a series of (3D)(1D) regions. For $n = 6$, we have a 4×4×4 (3D) region. That region contains one 2×1 (3D)(1D) region, which contains one family of (3D)(1D) normals. When $n = 7$, we have a 3×1 (3D)(1D) region with one family of (3D)(1D) normals, remembering not to count the center piece. When $n = 8$, we have a 4×1 (3D)(1D) region with two families, but also a 2×1 (3D)(1D) region within the (3D) region that contains one family: 1 + 2 = 3. When $n = 9$, we have a 5×1 (3D)(1D) region that contains two families, and a 3×1 (3D)(1D) region beneath it that contains one family: 1 + 2 = 3. When $n = 10$, we have a 6×1 region with three families, a 4×1 region with two families, and a 2×1 region with one family, 1 + 2 + 3 = 6. We are clearly dealing with the triangular numbers again, and in fact with the same counts as the (2D) wings, as can be seen by comparing these numbers with the table listed above for those pieces. Therefore, by the same reasoning the count for the number of families is $\lfloor (n-4)/2 \rfloor \lfloor (n-2)/2 \rfloor / 2$, making the number of configurations of the (3D)(1D) normals:

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{\left\lfloor \frac{n-4}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor}{2}}$$

Now we will consider the (3D)(2D) normals and (3D)(2D) wings. Observe that we can simplify things by counting both at once, since they each have the same permutation and orientation counts, and occur on the same set of (3D)(2D) regions.

Similarly to the (3D)(1D) normals and their associated regions, we must consider nested series of (3D)(2D) regions. Here we will display a series of diagrams of these regions, starting with $n = 6$. The regions will be shown in full, as a series: First, the outermost $(n - 4)$×$(n - 4)$ (3D)(2D) region, followed by the $(n - 6)$×$(n - 6)$ region beneath it, and continuing until we end with a 2×2 or 3×3 region. Here are the diagrams, with the (3D)(2D) normals and wings representing their family being marked with an x:

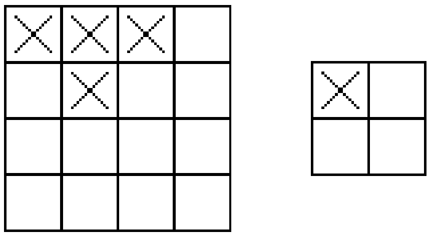
$n = 6$:

X	

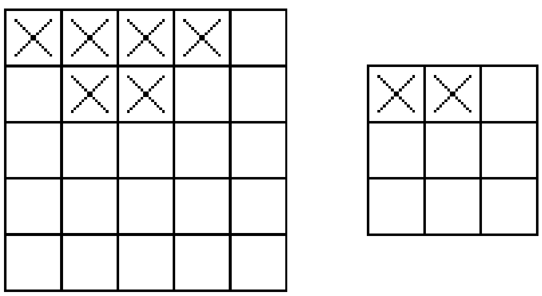
$n = 7$:

X	X	

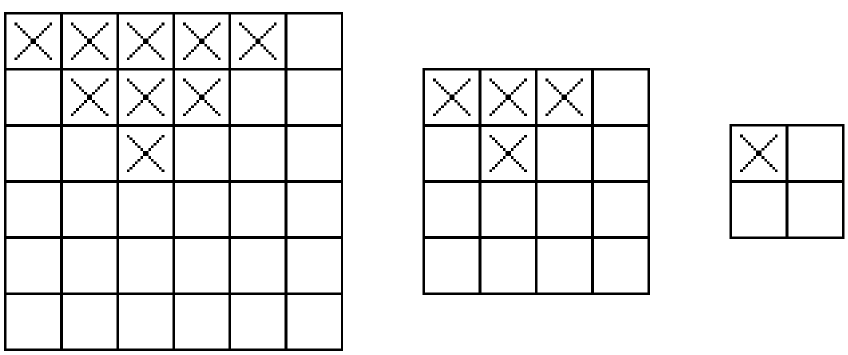
n = 8:



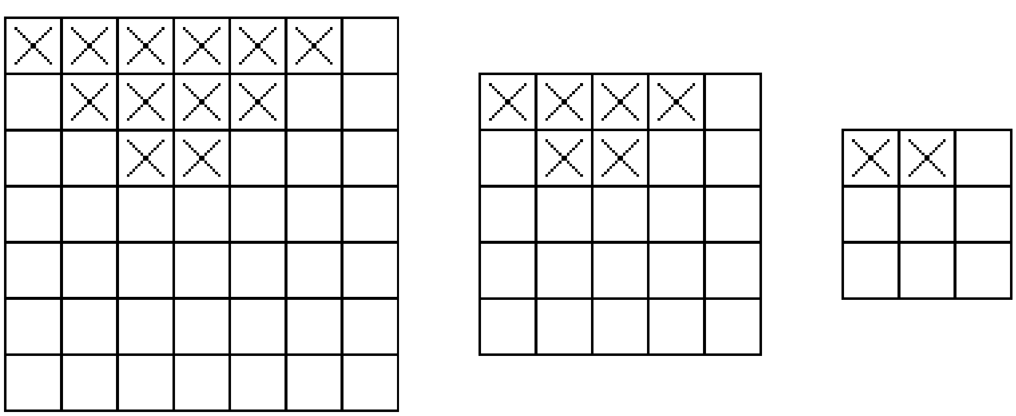
n = 9:



n = 10:



n = 11:



We will have to consider the cases of odd and even cubes separately. First, odd cubes.

We can identify a pattern; it is a sum of sums of consecutive even numbers. Here we list the pattern for $7 \leq n \leq 15$:

n = 7: 2
n = 9: 2 + (2 + 4) = 8
n = 11: 2 + (2 + 4) + (2 + 4 + 6) = 20
n = 13: 2 + (2 + 4) + (2 + 4 + 6) + (2 + 4 + 6 + 8) = 40
n = 15: 2 + (2 + 4) + (2 + 4 + 6) + (2 + 4 + 6 + 8) + (2 + 4 + 6 + 8 + 10) = 70

The numbers continue: 112, 168, 240, ...

It would be difficult to derive a closed form for this sequence by hand. Fortunately, there is a very useful website known as The On-Line Encyclopedia of Integer Sequences.[6] Entering our sequence into the website we obtain one result, a sequence with the following formula:

$$2\binom{m}{3}$$

where $\binom{m}{3}$ is a binomial coefficient, equal to m(m - 1)(m - 2)/6.

This formula needs to be modified, as it produces the sequence 2, 8, 20, 40, 70, ... for m equal to 3, 4, 5, 6, 7, ... respectively. Here is a table listing corresponding values of m and n for $7 \leq n \leq 15$:

n = 7: m = 3
n = 9: m = 4
n = 11: m = 5
n = 13: m = 6
n = 15: m = 7

We can see that m = (n - 1)/2, therefore our formula becomes

$$2\binom{\frac{n-1}{2}}{3}$$

which is equivalent to 2((n - 1)/2)((n - 3)/2)((n - 5)/2)/6, which equals:

$$\frac{(n-5)(n-3)(n-1)}{24}$$

Although this formula appears correct as the only sequence listed in The Encyclopedia, to be complete we must prove it is correct. To do this we will use a slightly modified proof by induction. We will first prove it correct for n = 7, then show that correctness for n = k implies correctness for n = k + 2.

The formula is correct for n = 7; (7 - 5)(7 - 3)(7 - 1)/24 = 2, which is the first term in our sequence above.

Now we assume that it holds for n = k, that is,

$$\frac{(k-5)(k-3)(k-1)}{24} = 2 + (2+4) + (2+4+6) + ... + (2+4+ ... + (k-5))$$

Assuming this, we write an expression for n = k + 2:

$$2 + (2+4) + (2+4+6) + ... + (2+4+ ... + (k-3))$$

We must show that this is equal to

$$\frac{((k+2)-5)((k+2)-3)((k+2)-1)}{24} = \frac{(k-3)(k-1)(k+1)}{24}$$

Using our equation above, we rewrite our expression for n = k + 2 as:

$$\frac{(k-5)(k-3)(k-1)}{24} + (2+4+ ... + (k-3))$$

The latter term is equal to 2(1 + 2 + ... + (k - 3)/2), which equals 2((k - 3)/2)((k - 3)/2 + 1)/2, by the triangular numbers formula, which in turn simplifies to (k - 3)(k - 1)/4. Substituting this into our expression gives:

$$\frac{(k-5)(k-3)(k-1)}{24} + \frac{(k-3)(k-1)}{4}$$

which equals

$$\frac{(k-5)(k-3)(k-1) + 6(k-3)(k-1)}{24}$$

This simplifies to

$$\frac{(k-3)(k-1)(k-5+6)}{24} = \frac{(k-3)(k-1)(k+1)}{24}$$

and thus we have proven our formula correct.

Therefore, we have shown that for odd n, the number of configurations of the (3D)(2D) normals and wings is

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{(n-5)(n-3)(n-1)}{24}}$$

which can be written for all n as:

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{(n \bmod 2)(n-5)(n-3)(n-1)}{24}}$$

This expression is correct for n < 7, so it does not need to be modified.

Now we will examine the (3D)(2D) normals and wings for even n.

We have a similar pattern, a sum of sums of consecutive odd numbers. Here is the pattern for $6 \le n \le 14$:

n = 6: 1
n = 8: 1 + (1 + 3) = 5
n = 10: 1 + (1 + 3) + (1 + 3 + 5) = 14
n = 12: 1 + (1 + 3) + (1 + 3 + 5) + (1 + 3 + 5 + 7) = 30
n = 14: 1 + (1 + 3) + (1 + 3 + 5) + (1 + 3 + 5 + 7) + (1 + 3 + 5 + 7 + 9) = 55

which continues 91, 140, 204, ...

The formula for this sequence according to The Encyclopedia is:

m(m + 1)(2m + 1)/6

which produces the values above for m equal to 1, 2, 3, 4, 5, etc. Here we have a table for corresponding values of m and n, $6 \le n \le 14$:

n = 6: m = 1
n = 8: m = 2
n = 10: m = 3
n = 12: m = 4
n = 14: m = 5

We observe that m = (n - 4)/2, this makes our formula ((n - 4)/2)((n - 4)/2 + 1)(n - 4 + 1)/6, which equals:

$$\frac{(n-4)(n-3)(n-2)}{24}$$

Now we must prove that this formula is correct, using the same method of proof by induction as before. We first show it is correct for n = 6, then prove that correctness for n = k implies correctness for n = k + 2.

The formula is correct for n = 6, as (6 - 4)(6 - 3)(6 - 2)/24 = 1. We now assume that our formula holds for n = k:

$$\frac{(k-4)(k-3)(k-2)}{24} = 1 + (1+3) + (1+3+5) + ... + (1+3+ ... + (k-5))$$

Here we have an expression for n = k + 2:

$$1 + (1+3) + (1+3+5) + ... + (1+3+ ... + (k-3))$$

We must prove that this is equivalent to

$$\frac{((k+2)-4)((k+2)-3)((k+2)-2)}{24} = \frac{(k-2)(k-1)k}{24}$$

Using our assumption, we can rewrite the expression for n = k + 2 as:

$$\frac{(k-4)(k-3)(k-2)}{24} + (1+3+\ldots+(k-3))$$

It is known that that sum of the first p odd natural numbers is equal to p^2. Since there are ((k - 3) + 1)/2 = (k - 2)/2 odd numbers less than or equal to k - 3, we can simplify our expression to:

$$\frac{(k-4)(k-3)(k-2)}{24} + \frac{(k-2)^2}{4}$$

This equals

$$\frac{(k-4)(k-3)(k-2) + 6(k-2)^2}{24} = \frac{(k-2)[(k-4)(k-3) + 6(k-2)]}{24} =$$

$$\frac{(k-2)\left(k^2 - 7k + 12 + 6k - 12\right)}{24}$$

which simplifies to

$$\frac{(k-2)\left(k^2 - k\right)}{24} = \frac{(k-2)(k-1)k}{24}$$

as desired.

We have therefore shown that for even n, the number of configurations of the (3D)(2D) normals and wings is

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{(n-4)(n-3)(n-2)}{24}}$$

This can be written for all n using a combination of the modulo operation and the absolute value function: $|(n \mathbf{mod}\ 2) - 1|$ equals 1 when n is even, and 0 when n is odd. Therefore, our calculation can be written as

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{|(n \mathbf{mod}\ 2) - 1|(n-4)(n-3)(n-2)}{24}}$$

which is correct for n < 6.

We can now combine our three results for the number of families of the (3D)(1D) normals, (3D)(2D) normals, and (3D)(2D) wings. We multiply them together to obtain:

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{\left\lfloor \frac{n-4}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor}{2} + \frac{(n \mathbf{mod}\ 2)(n-5)(n-3)(n-1) + |(n \mathbf{mod}\ 2) - 1|(n-4)(n-3)(n-2)}{24}}$$

as our count of the number of configurations of the (3D)(1D) normals, (3D)(2D) normals, and (3D)(2D) wings.

Step 9: The Formula

Having found the number of configurations of each type of piece, our last step is to multiply all of them together, obtaining $C_4(n)$, the number of configurations of an n×n×n×n Rubik's Cube:

$$C_4(n) = \frac{15!\cdot 12^{15}}{6}\left(24!\cdot 32!\cdot 2^{26}\cdot 6^{33}\right)^{n \,\mathbf{mod}\, 2}\left(\frac{64!}{2}\cdot 3^{63}\right)^{\left\lfloor \frac{n-2}{2}\right\rfloor}\left(\frac{96!}{24^{24}}\cdot 2^{95}\right)^{\left\lfloor \frac{n-2}{2}\right\rfloor + (n \,\mathbf{mod}\, 2)\left(\frac{n-3}{2}\right)}\cdot$$

$$\left(\frac{192!}{24^{48}}\right)^{\frac{\left\lfloor \frac{n-4}{2}\right\rfloor\left\lfloor \frac{n-2}{2}\right\rfloor}{2}}\left(\frac{64!}{(8!)^8}\right)^{\left\lfloor \frac{n-2}{2}\right\rfloor}\left(\frac{96!}{(12!)^8}\right)^{(n \,\mathbf{mod}\, 2)\left(\frac{n-3}{2}\right)}\left(\frac{48!}{(6!)^8}\right)^{(n \,\mathbf{mod}\, 2)\left(\frac{n-3}{2}\right)}\cdot$$

$$\left(\frac{192!}{(24!)^8}\right)^{\frac{\left\lfloor \frac{n-4}{2}\right\rfloor\left\lfloor \frac{n-2}{2}\right\rfloor}{2} + \frac{(n \,\mathbf{mod}\, 2)(n-5)(n-3)(n-1)\,+\,|(n \,\mathbf{mod}\, 2)\,-1|(n-4)(n-3)(n-2)}{24}}$$

12. Conclusion

It is hoped that the reader has enjoyed this journey, and that they have gained a deeper understanding of higher-dimensional Rubik's Cubes. Feel free to send comments, questions, suggestions, and corrections to this email address:

djs314djs314@yahoo.com

There will be sequels to this paper, the first of which will derive $C_5(n)$, the number of configurations of an n^5 Rubik's Cube. I would like to once again thank Melinda Green, Don Hatch, and Jay Berkenbilt for creating Magic Cube 4D, H. J. Kamack, T. R. Keane, and Eric Balandraud for their previous work, and especially Roice Nelson for his support, assistance, and inspiration.

13. References

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