A Derivation of the Number of Different Positions of the Fully-Colored Magic120Cell

By David Smith

1. Introduction

Magic120Cell is a computer simulation of a 4-dimensional permutation puzzle created by Roice Nelson.[1] It can be thought of as the 4D version of the Megaminx (a puzzle similar to the Rubik's Cube in the shape of a dodecahedron) because it is based on the 120-cell, one of the six regular polychora, which has 120 dodecahedral faces.[2] This paper will derive an upper bound for the number of legally obtainable positions of the fully-colored version of the program, in which each face is a different color. Thanks to a set of eight algorithms found collectively by Roice Nelson and Matthew Galla, this number has now been verified to be the exact count of the number of different positions, and the author is very much indebted to them for their efforts. A link to the algorithms is provided at the end of this paper. Only basic knowledge of group theory, combinatorics, Rubik's Cube notation, and the concept of higher dimensions is required by the reader. For more information on the 120-cell and Magic120Cell, please see the references below.

2. Explanation

In this section we will explain how the upper bound was derived. Along the way, it will be shown how the algorithms found by Roice Nelson and Matthew Galla establish that the upper bound is indeed an exact count.

Here is the number of positions:

$$\frac{600!}{2} \cdot \frac{1200!}{2} \cdot \frac{720!}{2} \cdot \frac{2^{720}}{2} \cdot \frac{6^{1200}}{2} \cdot \frac{12^{600}}{3}$$

The exact value was computed using the free computer algebra system Yacas;[3] here it is in its full glory:

23435018363697222779126210606140343600982219866708667227704291465940007
37743198001537086016413748065359228217622633869330769129523601891497799
90823414733250819032377663096727895392891107724676361939174468537213471
84699260131924584724938945790242680862147295113762851571432130901040238
96149551266842769465158629370618815995041314328829732432057176063611161
23422302770133676753359134856348612503635674252607065815753807941966366
98057536512196715919594180779891338303538085006270891583549467992567391
85180535778985103137974951114346934416286264525322532242698044327455362
45594013789336900464999769975314632465421639791307831594564938014806846

43170816415770010483963217284920963354265992129473309221874222731561178
16542353296798264919536869373254412130565886195935986667368898349834998
21329543130389608025077440970771216857898162084209764122888617411552472
35969553038560987694512525780640891845410615026444483377179855326132898
75234261959552618641928258383934632570287455387991780347467121010722113
61928836844443707162412473785444967682885281547729595860180055748863425
82987146883285105106538113133816701062677558383952546932927579065352378
01699938857635611816907866063280477566711511600651402621287007177419657
47137395706297269591116929204261763967322064643743204180740840609622274
50477533328851963152796037024975768039218238701900252954269938177351575
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77228966566309226326618668840915289587325249776450099751298794240127365
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30327207836333881250393819594360829900032177090494274930448392889865527
16614065052187986459391365691818487107035801789611790557625739664732160
65851938727927023709296602229334790711981400149187774330090686038188693
91261606235286494542181170865114851849953373717543236061723434256780261
70547107650201457180103595669061215322965129698879686174938686442023883
00748340309101390837462739717879861015966407725537701315760595302551716
71986415959726651855198833270638521668923163972072488967633862968020464
59857154591489337994834077319666200102594473324166310981151194815089205
57151182894539191982468293894472660275087710846277134524865818266133054
42334380903271185037626999712118120957184637997454373033536708899691932
67133888402013848180589600375369501541798982850283307728992349369110105
46520467826426004880473115209607600645972315535718172987982451474844986
38207939550826131991321502334364118044702920268720341762396367094618866
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14850013029742851991570001253742052023463664478943640221712458729805575
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44262379730778251303396955855695117379535034691606466408011495289375314
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82557352902277742867710678069168798437309117460470080742233210420451923
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85434125488705753096939302756128054596976405648147286686571256428614441
70150339281973567744826847917503807378348535977716210565251241466128121
27489744510755607030104354706835298463258585993214964912284096487358233
65511645083418224779444054273548541176504221269485074946310915002488718
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91464434261509507888671511637033790932545587351156241243113539812047803
32568991232059877343852662991568615775896662637181748468409947337080580
40572024450182896514659885925637625707421001762988412973222775700881854
24686720928281296902412091261171945310151555961276870291126372512436270
06952830074271589382737534210078713204605113743280142889267219197170354
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75841278684502282053918624209180100149726514306210673113609117091246135
65223035254373475279222721857792173328260547996939875237576837159815118
68437995628528234082912911481312451144148067422323644080656449490476178

69974972208660720805312515123592330897469991534040275425615810166559862
90453613626399014292854794414276344511148330328685197189376868495672292
92219730863040706516534661225522594916193135982360752630702024083643694
98309370698498093084824000430278614244379613737044583505562135097992678
05260822568001712636170958894341958120566294218192269192589063843887982
76399691863692115560126140715610109114003149849449412445547918396053785
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74641197649400613289762440747999594711818677478380093322693390443497615
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24442974862512599608887589752218559193144931596281284315382618742792620
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90715982378923238819174377671246115948128333247228319128499096287090364
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08811101917211049364489819890827973553466812312700794247970136249599713
68830975248367892523082396738072274831267273049793679458450960225575330
90840325055925127369491405078011569600959829055592354981900265212992888
74537523085955044911868546931655826676111114140917917721449373043059908
24075969774780698659600903247322382509271117981454345778343923721701140
34040387142730946291948768518544291460594918104272439297270660195239204
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29742915483006071654290990776376332377597123229369363319211034520828156
16383626599751692734054125142693424208441259140739967321942103460390485
73512549204538199361441602981588922796564372729802637150967463996220269
92509662606254579651749991204772662937610983604733514590588466763484779
75336521786978901110936704729127455396942554264720541493172351367586278
52118009553781736752460941012653895714556808971968820220233708185524289
26324734529251413367934964381909880343066993726638347012446562279909471
00665871099287936575368913555297252102692185719691751527961183922455231
72371787084227168597930766375481134730976264840880937562376002100356262
35107696623982892463214959186113390887996406714662349935032809366747705
65928739057221107528446966775483155772142995330294320084551917275407602
87830218813852897349462068161826723496382625465167461901840049431850524
89018407136301332421978685188429040584176333209422917640992438642326814
48471988797114061727140678598275664209888253021405519815632168746508451
06268660985414101246860290152701992710734632469139060273152361598118124
26751417487110100479881455904096718779749277515897027333976945734065218
13427043475236100473238801150143720068671986307968791851735260237830993
59283951655340545557118534217560207955199404424096337283839953943573882
77230454238664055061747285720327136114251359554586129278357158728391085
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or approximately 2.3×10^{8126}. What a number!

Now for the explanation. First of all, Magic120Cell has 120 1-colored immobile center pieces. It also has 720 2-colored pieces, because each dodecahedral face contains 12 2-colored pieces and each piece lies in two faces, giving $(120 \cdot 12)/2 = 720$. Similarly, there are 1200 3-colored pieces and 600 4-colored pieces. Note that because the 1-colored pieces are immobile, we can calculate the number of positions by counting the number of ways each other type of piece can be permuted and oriented around them.

We now consider what takes place when we rotate a dodecahedral face 72 degrees about a line connecting the midpoints of two opposite pentagonal faces. Note that we never need to consider other types of rotations because any possible configuration can be reached from the solved state by using only the 72 degree face rotations mentioned above. In such a rotation, we observe that there are two 5-cycles of the 2-colored pieces, six 5-cycles of the 3-colored pieces, and four 5-cycles of the 4-colored pieces. Since 5-cycles are even permutations, all permutations of each type of piece, and also for the facelets of each type of piece, are even.

Therefore, the 720 2-colored pieces can be permuted in 720!/2 ways, dividing by two beacuse of the even parity. Similarly, the 3-colored pieces and 4-colored pieces can be permuted 1200!/2 and 600!/2 ways, respectively. Multiplying these three counts together, we obtain an upper bound for the number of ways the pieces of Magic120Cell can be permuted without regards to orientation:

$$\frac{600!}{2} \cdot \frac{1200!}{2} \cdot \frac{720!}{2}$$

Roice Nelson has found three algorithms which show that every permutation described above is actually reachable in Magic120Cell. The first performs a single 3-cycle of the 2-colored pieces, but disregards preserving the state of the 3-colored and 4-colored pieces. The second algorithm performs a single 3-cycle of the 3-colored pieces, while preserving the state of the 2-colored pieces but not the state of the 4-colored pieces. The final algorithm performs a single 3-cycle of the 4-colored pieces without affecting any other pieces. It can be seen that these three algorithms, when applied in the order listed above, can produce any permutation of pieces in which the parity of each type of piece is necessarily even.

Now for the orientations. Since the facelets also undergo even permutations, all orientations will be limited to even permutations of facelets.

Therefore the first 719 2-colored pieces can be oriented in 2 ways each, but the last will be determined by the others because of the even parity. This results in

$$\frac{2^{720}}{2}$$

ways of orienting the 2-colored pieces. Matthew Galla has provided an algorithm which flips the orientation of two 2-colored pieces without affecting any other pieces, thus proving that any orientation of the 2-colored pieces which is of even parity can be achieved.

For the 3-colored and 4-colored pieces, I followed closely the methods of T. R. Keane and H. J. Kamack in their paper, "The Rubik Tesseract",[4] modifying their arguments as necessary to apply them to

Magic120Cell.

Any 3-colored piece can be oriented in 6 different ways. (not three, because in four dimensions we can reflect 3-colored pieces as well as twist them!) Notice that a twist (a 3-cycle of the facelets on that piece) is an even permutation, while a reflection (a 2-cycle of two of the facelets on that piece) is an odd permutation. Since the total parity of all of the 3-colored pieces must be even, the first 1199 3-colored pieces can be oriented in 6 ways each, while the last can be oriented in only 3. (If the first 1199 3-colored pieces total to an even permutation, the last 3-colored piece must be one of the 3 even twists, while if they total to an odd permutation, the last 3-colored piece must be one of the 3 odd reflections.) This gives a total of

$$\frac{6^{1200}}{2}$$

ways of orienting the 3-colored pieces. Two algorithms have been found by Matthew Galla which show that this count is exact. One of them twists one 3-colored piece without affecting any of the others, and the other reflects two 3-colored pieces without affecting any others.

Finally, the toughest part. The orientation of the 4-colored pieces required a generalization of the group theory based solution for the corners of the 3^4 cube discovered by Keane and Kamack to Magic120Cell.

In their paper, Keane and Kamack first describe that there are 24 permutations of the facelets of a 4-colored piece, comprising the S_4 group (The symmetric group on four letters.) They describe orientations using cycle notation of the four facelets, labeled a, b, c, and d.

The 24 different orientations can be broken down into four crosses, (ab)(cd), (ac)(bd), (ad)(bc), and I, the identity; eight 3-cycles, called twists; six 2-cycles; and six 4-cycles. However, for corner pieces on a cube of any dimension, only even permutations of the facelets can occur, because the odd permutations are mirror images. Thus, 2-cycles and 4-cycles cannot occur because they are odd permutations.

Hence, each 4-colored piece can only be oriented in 12 ways. The even permutations are all possible, and form the alternating group A_4. Keane and Kamack continue by observing that the crosses are a normal subgroup of the alternating group they call N. So,

N = {I, (ab)(cd), (ac)(bd), (ad)(bc)}

The cosets of N consist of the twists, which they call S and Z:

S = {(abc), (adb), (acd), (bdc)}

Z = {(acb), (abd), (adc), (bcd)}

They then note that the sets N, S, and Z form the quotient-group of A_4 by N, in which N acts as the identity:

$$\frac{A_4}{N} = \{N, S, Z\}$$

We can then see that the group multiplication table is:

	N	S	Z
N	N	S	Z
S	S	Z	N
Z	Z	N	S

From this table, we can see that this quotient-group is isomorphic to the group of residue classes, mod 3. This means that we can assign the number 0 to N, the number 1 to S, and the number -1 to Z, and adding these numbers mod 3 is the same as taking the products of elements of these three subgroups.

Notice that this entire argument applies equally well to Magic120Cell as to the tesseract. Now the only thing left to show is that the sum of the orientations of the 4-colored pieces (counting 0 for an orientation in N, etc.) mod 3 is always the same, whether to the tesseract or Magic120Cell.

The orientations can be defined by assigning, to each 4-colored piece, a letter to each facelet and each position of each facelet. Then each orientation can be described by a 4-letter string (e.g. ABCD) relative to the position it is occupying.

When pieces or cubies rotate in a cycle, their facelets undergo n disjoint cycles if they have n facelets. In the tesseract, every corner rotation boils down to four 4-cycles of facelets for each cycle of four cubies. In Magic120Cell, it is the same except it is four 5-cycles. If we can show that in cycles of any length, the sum of the orientations of the pieces mod 3 does not change, we will have proved this for both the tesseract and Magic120Cell.

Consider a 2-cycle of 4-colored pieces:

ABCD 1
ABCD 2

Each row represents a 4-colored piece. The 2-cycles of facelets are vertical in direction. For example:

ABCD 1
CDAB 2

This means that facelet A on piece 1 goes where facelet C on piece 2 was, etc. In this example, piece 1 performed an N-twist. Now notice that since we are dealing with cycles, the facelets of piece 2 must return to the original positions of the facelets of piece 1. Therefore, piece 2 also performed an N-twist. It can be checked that if piece 1 performs a Z-twist, piece 2 performs an S-twist, and if piece 1 performs an S-twist, piece 2 performs a Z-twist. Therefore, the sum of the values mod 3 does not change, and equals zero.

Now we simply note that a cycle of pieces of any length can always be expressed as a product of 2-cycles, implying that the sum of the orientation values mod 3 equals zero regardless of the length of the cycles involved. It follows that this is true for the corners of the 2^4 cube as a special case. Since an N-twist is 0, we can have an isolated N-twist without affecting any other pieces. The value S - Z must therefore be congruent to zero, mod 3.

This means that the first 599 4-colored pieces can each be in any of 12 orientations. If the value of orientations up to that point is 0, the remaining value must be an N-twist. If it is 1, the remaining value must be a Z-twist, and if it is -1, the remaining value must be an S-twist. In each case there are four possible orientations left for the last 4-colored piece. Therefore, the upper bound for the orientations of the 4-colored pieces is

$$\frac{12^{600}}{3}$$

In order to show that this value is exact, algorithms are needed which show that any N-twist can be performed without affecting the rest of the pieces, and that any Z-twist can be performed along with any S-twist without affecting any other pieces. Matthew Galla has found two algorithms which do just that. The first is an N-twist of the form (ab)(cd). Note that this twist can represent any of the three non-identity N-twists by means of symmetry: We can simply relabel the facelets we are dealing with, since all of the non-identity N-twists are essentially of the same form.

The second algorithm is a Z-twist and S-twist pair of the form (abc)(acb). Call this move A, and the N-twist (ab)(cd) B. It can be seen that by combining A and B, we can generate all possible pairs of S-twists and Z-twists, and thus all of the orientations of the 4-colored pieces that can be produced in Magic120Cell are contained in my count above. (See "The Rubik Tesseract", Appendix A, for a complete description of exactly how this is achieved.)

When we multiply these orientation counts with the one found for permutations, we arrive at the number listed above for the number of different positions of the fully-colored Magic120Cell.

The author would like to thank Roice Nelson and Matthew Galla for contributing the necessary algorithms to verify my upper bound to be an exact count. Also, he would like to thank Roice Nelson with the deepest gratitude for creating Magic120Cell, which without, he would have never had the inspiration to perform this calculation.

3. Algorithms

Here is a link to the algorithms discovered by Roice Nelson and Matthew Galla, ready to be loaded into Magic120Cell. To view the algorithms, you can use the Undo and Redo commands, or the Solve command to view them in reverse.

www.gravitation3d.com/david/M120C_algs.rar

4. References

[1] Nelson, R., Magic120Cell, http://www.gravitation3d.com/magic120cell

[2] Wikipedia, http://en.wikipedia.org/wiki/120-cell

[3] Yacas, http://yacas.sourceforge.net

[4] Kamack, H. J. and Keane, T. R. (1982), "The Rubik Tesseract," Newark